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QUANTISATION DEFORMS w_∞ TO W_∞ GRAVITY

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Quantising a classical theory of w_∞ gravity requires the introduction of an infinite number of counterterms in order to remove matter-dependent anomalies. We show that these counterterms correspond precisely to a renormalisation of the classical w_∞ currents to quantum W_∞ currents.

1. Introduction

In the light of the current interest in two-dimensional quantum gravity, with its underlying Virasoro symmetry, it is natural to consider possible generalisations in which the Virasoro algebra is extended to a higher-spin conformal algebra. One motivation for such generalisations is the possibility of incorporating an infinite number of Virasoro primary fields into a finite set of representations of the enlarged algebra; in particular, this may enable one to construct rational conformal field theories for $c > 1$. Amongst the possible generalisations are the W_N algebras [1,2], and their $N \to \infty$ limits [3–5]. The finite-N W_N algebras contain generators with conformal spins $2, 3, \ldots, N$; closure is achieved at the expense of having

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non-linearity in the algebra. In the $N \to \infty$ limit, one can regain linearity in a number of ways.

The first $N = \infty$ algebra to be discovered [3], called w_∞, can be viewed as the algebra of symplectic diffeomorphisms of a cylinder [3,5]. The structure of w_∞ is very simple:

$$\left[v^i_m, v^j_n\right] = ((j+1)m - (i+1)n)v^{i+j}_{m+n}, \tag{1.1}$$

where v^i_m denotes the mth Fourier mode of the spin-$(i+2)$ current $v^i(z)$. One can easily verify that the Jacobi identities permit only the usual Virasoro central term in this algebra. In order to have the possibility of a richer central extension, in which diagonal central terms can occur for all conformal spins, one must look at solutions of the Jacobi identities for more general algebras than eq. (1.1). For the given set of fields with spins $2, 3, \ldots, \infty$, there is a unique such algebra, which is called W_∞ [5]. The form of the algebra is

$$\left[V^i_m, V^j_n\right] = \sum_{r \geqslant 0} g^{ij}_{2r}(m,n) V^{i+j-2r}_{m+n} + c_i(m)\delta^{ij}\delta_{m+n,0}. \tag{1.2}$$

The details of the structure constants $g^{ij}_r(m,n)$ and the central terms $c_i(m)$ may be found in ref. [5]. In the case of both w_∞ and W_∞, one can enlarge the algebra by adjoining a spin-1 current; the enlarged algebras are called $w_{1+\infty}$ and $W_{1+\infty}$ respectively. In the case of $w_{1+\infty}$, one simply extends the range of the indices i and j in eq. (1.1) to run from -1 to ∞ rather than 0 to ∞. For $W_{1+\infty}$, the construction is more complicated; both the structure constants and central charges receive modifications [6].

The w_∞ and $w_{1+\infty}$ algebras may be obtained as contractions of W_∞ and $W_{1+\infty}$ respectively [5,6], by scaling the generators according to

$$V^i_m = q^{-i} v^i_m, \tag{1.3}$$

and sending the parameter q to zero. A natural speculation has been that the w-algebras might be properly viewed as classical limits of the W-algebras, with q related to Planck's constant $\hbar$. In this paper we shall show the precise sense in which this happens in the quantisation of scalar-matter realisations of W-symmetries.

The gauge theories of W-algebras are called W-gravity theories. They have been studied extensively in the last few years [7–12]. Our starting point is the gauge theory of w_∞ gravity [9], consisting of scalar fields that provide a non-linear realisation of w_∞, coupled to background gauge fields for each spin. In ref. [9] both chiral and non-chiral gaugings were considered, at the classical level. Here, we shall restrict attention to the chiral gauging, and discuss the quantisation of the theory.

Several recent papers have addressed the issues of quantisation and anomalies for chiral W_3 and W_N gravities [13–19]. In these cases the analysis is complicated by the fact that not only the realisations, but also the algebras themselves, are non-linear. There are two categories of anomalies that can arise in such theories. The first, called universal anomalies, are given by local expressions involving the background gauge fields only. The second consists of matter-field-dependent anomalies, which arise from diagrams with external matter fields. The possibility of such anomalies generally arises in theories with non-linearly realised symmetries. A key point here is that although there exist "quantum" operator-product realisations of W_N [1,2], these involve normal ordering with respect to the modes of the *currents* and not the modes of the matter *fields* of the realisation. This type of normal ordering does not enable one to evaluate arbitrary Green functions [15]. If one rewrites the operator-product relations for W_N using the field-mode normal ordering necessary for the construction of general Green functions, the algebra fails to close on W_N [15]. Apparently quantum W_N gravity, at least with the realisations considered thus far, suffers from anomalies for which no mechanism, to our knowledge, exists for their removal*. These matter-dependent anomalies cannot be ignored by looking at diagrams with external gauge fields only, since their occurrence in subdivergences will lead to non-local anomalies in higher-order diagrams with external gauge fields. Indeed the occurrence of such non-local anomalies has been shown in ref. [18]. It would be interesting to see whether, by using other realisations of the W_N symmetry, it might be possible to construct a consistent quantum theory of W_N gravity.

One way to overcome the difficulties described above is to remain within the confines of the normal ordering with respect to fields, and then to enlarge the set of currents in order to obtain an enlarged algebra that does close [17]. This approach cannot fail, in the sense that by continuing to include new currents as the need arises, one will eventually be able to interpret the result of any operator-product relation in terms of currents in the enlarged algebra. If this procedure is to be useful, one should be able to carry it out in a systematic manner, arriving at an algebra that respects some group-theoretic organising principle.

In this paper, we shall approach the problem from a somewhat different angle. Rather than enlarging an algebra such as W_N, we shall begin by considering the quantisation of the classical chiral w_∞ gravity theory constructed in ref. [9]. The currents in this classical realisation take the form

$$V^i = \frac{1}{i+2}\,\mathrm{tr}(\partial\varphi)^{i+2}, \tag{1.4}$$

where φ represents a scalar field, or a set of SU(N)-valued scalar fields. One can

* In a recent paper, the effect of including ghosts has been considered for W_3 gravity [19]. Apparently, however, this is not sufficient to cancel all anomalies.

also restrict attention to the $N-1$ scalar fields in the Cartan subalgebra. These currents, of spins $(i+2)$, generate the w_∞ algebra at the classical level, i.e. at the level of single contractions in the operator-product expansion. For simplicity, we shall concentrate on the single-scalar realisation.

At the full quantum level (i.e. multiple as well as single contractions), the currents (1.4) do not close to form an algebra. By computing loop diagrams in w_∞ gravity initially defined using these currents, we shall show how one can iteratively renormalise the currents and the gauge-transformation rules of the background gauge fields in order to eliminate all matter-dependent anomalies by the introduction of finite local counterterms. This is equivalent to adjusting the currents (1.4) by the addition of $\hbar$-dependent terms involving fewer φ-fields but with the same number of derivatives. In fact this provides a complementary and more elegant way of understanding the process of matter-anomaly cancellation. As we shall show, the modifications of the currents (1.4) necessary to make them close at the quantum level imply that they will now generate not the original w_∞ algebra, but precisely the W_∞ algebra. The w_∞ to W_∞ renormalisation procedure bears some similarity to the renormalisation of the supersymmetry algebra considered in ref. [20].

Having established that the renormalised quantum algebra is W_∞, we shall see that it is most convenient to discuss the universal anomalies by taking the renormalised W_∞ currents as our starting point for deriving anomalous Ward identities. In this way, we are guaranteed not to meet any matter-dependent anomalies, and thus we may focus attention on diagrams with external gauge-field lines only. The operator-product realisation of the W_∞ algebra allows us to derive an anomalous Ward identity to all loop orders, showing that the universal anomalies are local and are simply governed by the central-charge structure of the algebra.

In order to obtain a proper theory of W_∞ gravity, the universal anomalies need also to be cancelled. One way to do this is to construct a "critical" theory, in which the anomalies are simply cancelled against contributions from the W_∞ ghosts that arise when integrating over the W_∞ gauge fields. The single-scalar realisation that we are mainly considering in this paper has a specific background charge that implies a value $c=-2$ for the central charge. For this we require a ghost contribution of $+2$ in order to cancel the anomalies. Remarkably, this is precisely the (zeta-function regularised) value that one finds for the W_∞ ghosts [21,22]. An alternative way to handle the universal anomalies would be to introduce a "Liouville sector", by adding additional Weyl and associated higher-spin degrees of freedom, into whose classical symmetries the anomalies may be shifted. This would correspond to a non-critical theory of W_∞ gravity.

As was shown in ref. [9], the w_∞ gravity theory possesses an additional kind of symmetry, namely a Stueckelberg shift symmetry of the gauge fields. In the conclusion we shall discuss the fate of this symmetry at the quantum level.

2. Renormalising w_∞ gravity

The classical theory of chiral w_∞ gravity that will form our starting point is described by the action $S = 1/\pi \int d^2z L$, where L is given by [9]

$$L = \frac{1}{2}\bar{\partial}\varphi\,\partial\varphi - \sum_{i\geqslant 0}\frac{1}{i+2}A_i(\partial\varphi)^{i+2}, \tag{2.1}$$

where $\partial = \partial_z = \partial_-$ and $\bar{\partial} = \partial_{\bar{z}} = \partial_+$ (corresponding to a euclidean signature on the world-sheet). The action is invariant under the following local ($z, \bar{z}$ dependent) w_∞ transformations:

$$\delta\varphi = \sum_{l\geqslant 0} k_l(\partial\varphi)^{l+1}, \tag{2.2}$$

$$\delta A_l = \bar{\partial}k_l - \sum_{j=0}^{l}\big((j+1)A_j\partial k_{l-j} - (l-j+1)k_{l-j}\partial A_j\big). \tag{2.3}$$

This single-scalar action is also classically invariant under further Stueckelberg-type "shift" transformations of the gauge fields A_l [9] with $l \geqslant 1$, to whose quantum fate we shall return later.

We now proceed to quantise the above chiral w_∞ gravity theory. Our first concern will be to eliminate the matter-dependent anomalies from the theory by suitable finite renormalisations of the currents and transformations. The order parameter for the renormalisation programme will as usual be Planck's constant $\hbar$, which we shall write explicitly. We shall in fact find it necessary to expand in half-steps of $\sqrt{\hbar}$. It is worth emphasising that the theory defined by (2.1) has primitive divergences only in tadpole Feynman diagrams. This is because the $d = 2$ Lorentz invariance requires that the same number of $\bar{\partial}$-derivatives must appear in a local counterterm as there are positive Lorentz charges on the gauge fields in the counterterm (A_i caries Lorentz charge $i + 2$), and this makes the overall degree of divergence of all but the tadpole diagrams negative. The tadpole divergences may be subtracted from the theory by a normal-ordering prescription. As usual, we shall define the renormalised Green functions of the theory by normal ordering with respect to the modes of $\partial\varphi$, expanding this conformal field into a Laurent expansion in z^{-n-1} and ordering the modes in a product so that modes with larger values of n stand to the right of modes with smaller values of n. The propagator for our theory is given by

$$\langle\varphi(z,\bar{z})\varphi(w,\bar{w})\rangle = \hbar(\log(z-w) + \log(\bar{z}-\bar{w})). \tag{2.4}$$

As usual, the calculation of Feynman diagrams can be factorised into the separate

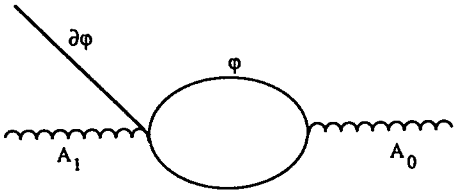

Fig. 1.

calculation of holomorphic and antiholomorphic parts. With the normal-ordering subtraction procedure, it may then be verified that all tadpole diagrams vanish.

The first diagram that can generate matter-dependent anomalies in the w_∞ algebra is given in fig. 1. The anomaly arising from this diagram has already been discussed in the W_3 analyses of refs. [17, 18]. It can be calculated by evaluating the double contractions in the operator product expansion of $1/(2\hbar)\int d^2 z A_0(z)(\partial\varphi(z))^2$ times $1/(3\hbar)\int d^2 w A_1(w)(\partial\varphi(w))^3$. The resulting contribution to the effective action is

$$\begin{aligned}\Gamma_{01\varphi} &= \frac{\hbar}{\pi^2}\int d^2z\, d^2w A_0(z) A_1(w) \frac{1}{(z-w)^4}\partial\varphi(w) \\ &= -\frac{\hbar}{6\pi}\int d^2z\, d^2w A_0(z) A_1(w) \frac{\partial_z^3}{\partial_{\bar z}}\delta^2(z-w)\,\partial\varphi(w) \\ &= -\frac{\hbar}{6\pi}\int d^2z \left(\frac{\partial^3}{\bar\partial}A_0(z)\right) A_1(z)\,\partial\varphi(z)\,. \end{aligned} \tag{2.5}$$

Under the leading-order inhomogeneous terms in the gauge transformations (2.3) ($\delta A_0 = \bar\partial k_0 + \dots$, $\delta A_1 = \bar\partial k_1 + \dots$) the anomalous variation of $\Gamma_{01\varphi}$ is

$$\delta\Gamma_{01\varphi} = -\frac{\hbar}{6\pi}\int d^2z \left(A_1 \partial^3 k_0 - k_1 \partial^3 A_0\right)\partial\varphi\,. \tag{2.6}$$

Note that in the derivation of this result one may drop terms proportional to the φ-field equation, since these cancel in the quantum Ward identity [23] against terms involving operator insertions of the φ-transformations into the relevant one-loop diagrams.

The anomalous variation (2.6) can be cancelled by adding the finite local counterterms $L_{1/2} + L_1$, given by

$$L_{1/2} = \tfrac{1}{2}\sqrt{\hbar}\left(A_0 \partial^2\varphi + A_1 \partial\varphi\,\partial^2\varphi\right), \tag{2.7}$$

$$L_1 = \tfrac{1}{12}\hbar A_1 \partial^3\varphi\,, \tag{2.8}$$

and by simultaneously correcting the φ-transformation (2.2) by the extra terms $\delta_{1/2}\varphi+\delta_1\varphi$ given by

$$\delta_{1/2}\varphi=-\tfrac{1}{2}\sqrt{\hbar}\,(\partial k_0+\partial k_1\,\partial\varphi)\,, \tag{2.9}$$

$$\delta_1\varphi=\tfrac{1}{12}\hbar\,\partial^2 k_1\,. \tag{2.10}$$

The appearance of half-integer powers of $\hbar$ may at first seem surprising. In fact, the resulting order $\sqrt{\hbar}$ changes to the variation of the effective action cancel out completely. The desired anomaly-cancelling terms in the variation of the effective action are of order $\hbar$, as one would expect for a one-loop anomaly. They arise in the pattern

$$\delta_0 L_1+\delta_{1/2}L_{1/2}+\delta_1 L_0\,. \tag{2.11}$$

These variations cancel the anomalies in (2.6) completely.

The occurrence of the counterterms (2.7) and (2.8) implies that the original spin-2 and spin-3 currents of the form (1.4) have received corrections, so that they now take the form

$$V^0=\tfrac{1}{2}(\partial\varphi)^2+\tfrac{1}{2}\sqrt{\hbar}\,\partial^2\varphi\,, \tag{2.12}$$

$$V^1=\tfrac{1}{3}(\partial\varphi)^3+\tfrac{1}{2}\sqrt{\hbar}\,\partial\varphi\,\partial^2\varphi+\tfrac{1}{12}\hbar\,\partial^3\varphi\,. \tag{2.13}$$

The transformation rules for the matter field φ, including the corrections (2.9) and (2.10), are precisely those that follow from the standard expression

$$\delta\varphi(w,\bar{w})=\hbar^{-1}\sum_{l\geqslant 0}\oint\frac{\mathrm{d}z}{2\pi i}k_l(z)V^l(z)\varphi(w,\bar{w})\,, \tag{2.14}$$

where V^i are now the modified currents, given for spin-2 and spin-3 by eqs. (2.12) and (2.13).

One can in principle proceed, by looking at higher-order diagrams with higher-spin external gauge fields, to determine the appropriate modifications to all the higher-spin currents that are needed in order to remove matter-dependent anomalies. At the same time, the transformation rules for the φ-field will require higher-spin modifications too. As in the sample diagram studied above, the modifications to the φ-variation will be precisely those that follow by substituting the modified currents into (2.14). There are further kinds of matter-dependent anomalies, of types that are not illustrated by the diagram in fig. 1, whose cancellation requires that the gauge-transformation rules (2.3) should also be modified. To build up the entire structure of the modifications to currents and transformation rules by these diagrammatic methods would clearly be a cumber-

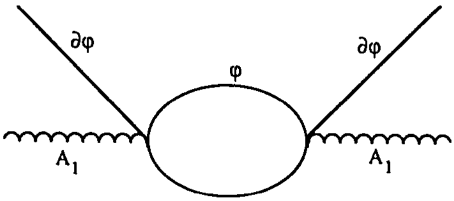

Fig. 2.

some procedure. Since we shall present a much simpler approach for deriving these results below, we shall for now just consider one more diagram, to illustrate the way in which the gauge-transformation rules (2.3) must receive corrections.

The simplest diagram that gives rise to a matter-dependent anomaly whose removal requires making modifications to the gauge-field transformation rules is shown in fig. 2. It produces a contribution to the effective action given by

$$\Gamma_{11\varphi\varphi} = -\frac{\hbar}{6\pi}\int d^2z A_1(z)\,\partial\varphi(z)\frac{\partial^3}{\bar{\partial}}\big(A_1(z)\,\partial\varphi(z)\big). \tag{2.15}$$

This gives rise to an anomalous variation with respect to the leading-order inhomogeneous term in the A_1 variation, i.e. $\delta A_1 = \bar{\partial}k_1 + \dots$:

$$\delta\Gamma_{11\varphi\varphi} = \frac{\hbar}{3\pi}\int d^2z A_1(z)\,\partial\varphi(z)\partial^3\big(k_1(z)\,\partial\varphi(z)\big). \tag{2.16}$$

Cancellation of this anomaly requires, in addition to the modifications to the spin-2 and spin-3 currents in eqs. (2.12) and (2.13) and the modifications (2.9) and (2.10) to the φ-transformation rules, a correction to the spin-2 gauge transformation rule:

$$\delta_1 A_0 = \tfrac{1}{20}\hbar\big(2\,\partial^3 A_1 k_1 - 3\,\partial^2 A_1\,\partial k_1 + 3\,\partial A_1\,\partial^2 k_1 - 2A_1\,\partial^3 k_1\big), \tag{2.17}$$

and a new counterterm

$$L_1 = \hbar A_2\Big(\tfrac{1}{5}\,\partial\varphi\,\partial^3\varphi - \tfrac{1}{20}\big(\partial^2\varphi\big)^2\Big). \tag{2.18}$$

This counterterm implies that the spin-4 current receives $O(\hbar)$ corrections. Since this is not the full story for the spin-4 current (other anomaly diagrams give rise to the need for further quantum corrections at different orders of $\hbar$), we shall defer giving the complete quantum-corrected spin-4 current until sect. 3.

We have now seen how the mechanism for cancelling the matter-dependent anomalies arising from the diagrams in figs. 1 and 2 leads to quantum corrections to the currents, and to the matter and gauge-field transformation rules. These constructions can be carried out to arbitrary order. Instead of continuing with the diagrammatic construction, in sect. 3 we shall show that the modifications to the lagrangian and transformation rules can all be understood as a renormalisation of w_∞ to W_∞.

3. The emergence of W_∞

The programme of anomaly cancellation via modification of the currents that we began in sect. 2 can be re-interpreted as a renormalisation of the currents that is necessary in order to achieve quantum closure of the operator-product algebra. The original currents (1.4) closed only classically, i.e. at the level of Poisson brackets, or, equivalently, at the level of single contractions in the operator-product expansion. At the quantum level, multiple contractions must also be taken into account.

In search of a closed quantum algebra, we begin by parametrising the renormalised currents as follows:

$$V^i = \frac{1}{i+2}(\partial\varphi)^{i+2} + \alpha_i\sqrt{\hbar}\,(\partial\varphi)^i\,\partial^2\varphi + \beta_i\hbar(\partial\varphi)^{i-1}\,\partial^3\varphi + \hbar\gamma_i(\partial\varphi)^{i-2}(\partial^2\varphi)^2 + \mathrm{O}(\hbar^{3/2})\,. \tag{3.1}$$

For now, to keep the discussion general, we shall allow a spin-1 current also, corresponding to taking $i=-1$. If we demand only that the algebra of these currents should close at the quantum level, we find, to the order that we are considering here, that the α_i coefficients are undetermined, and that the β_i and γ_i coefficients must satisfy the relation

$$\gamma_i - (i-1)\beta_i + \tfrac{1}{24}i(i-1)(i+1) = 0\,. \tag{3.2}$$

Obviously, there is an arbitrariness in the choice of the coefficients corresponding to the freedom to make redefinitions of the form $V^i \to V^i + \partial V^{i-1} + \dots$. In order to remove this arbitrariness, it is convenient to use SL(2, $\mathbb{R}$) covariance as an organising principle. This SL(2, $\mathbb{R}$) is generated by the -1, 0 and 1 Fourier modes of the spin-2 current V^0. Equivalently, it is generated by $\hbar^{-1}\oint(\mathrm{d}z/2\pi i)f(z)V^0(z)$, with $\partial^3 f(z)=0$. After requiring SL(2, $\mathbb{R}$) covariance, we find that only one arbitrary parameter, which we denote by α, remains. The coefficients α_i, β_i and γ_i are given by

$$\begin{aligned}
\alpha_i &= \alpha\,,\\
\beta_i &= \frac{i}{2i+1}\big(\alpha^2 + \tfrac{1}{12}(i-1)(i+1)\big)\,,\\
\gamma_i &= \frac{i(i-1)}{2i+1}\big(\alpha^2 - \tfrac{1}{8}(i+1)\big)\,.
\end{aligned} \tag{3.3}$$

Note that our results to this order give complete expressions for the spin-1, spin-2 and spin-3 currents.

The operator product expansion of the currents (3.1) then gives the result

$$
\begin{aligned}
V^i(z)V^j(w) \sim{}& \hbar(i+j+2)\frac{V^{i+j}}{(z-w)^2} + \hbar(i+1)\frac{\partial V^{i+j}}{z-w} \\
&+\frac{\hbar^2}{48}\left[1+\frac{3(1-16\alpha^2)}{(2i+1)(2j+1)(2i+2j+1)}\right] \\
&\times\left(6(i+j+1)(2i+2j+1)(i+j)\frac{V^{i+j-2}}{(z-w)^4}\right. \\
&+6i(i+j+1)(2i+2j+1)\frac{\partial V^{i+j-2}}{(z-w)^3} + 3i(i+j+1)(2i+1)\frac{\partial^2 V^{i+j-2}}{(z-w)^2} \\
&\left.+i(i+1)(2i+1)\frac{\partial^3 V^{i+j-2}}{(z-w)}\right) + \mathrm{O}(\hbar^3).
\end{aligned} \tag{3.4}
$$

The pattern found so far is clearly reminiscient of the structure of the $W_{1+\infty}$ algebra [5, 6]. To make this more precise, we shall give the complete result for the OPE of the spin-3 current with itself, since as we remarked above, our expression for the spin-3 current is complete. We find

$$
\begin{aligned}
V^1(z)V^1(w) \sim{}& 4\hbar\frac{V^2}{(z-w)^2} + 2\hbar\frac{\partial V^2}{z-w} \\
&+2\hbar^2(1-\alpha^2)\left(2\frac{V^0}{(z-w)^4} + \frac{\partial V^0}{(z-w)^3} + \frac{3}{10}\frac{\partial^2 V^0}{(z-w)^2} + \frac{1}{15}\frac{\partial^3 V^0}{(z-w)}\right).
\end{aligned} \tag{3.5}
$$

The set of OPEs between spin-1, spin-2 and spin-3 currents coincide exactly with those of the $W_{1+\infty}$ algebra [6] in the one-parameter family of bases described in refs. [24, 25]. (The parameter α is related to the parameter s in ref. [24] by $\alpha = s + \frac{1}{2}$, and to the equivalent parameter λ in ref. [25] by $\alpha = \frac{1}{2} - \lambda$.) Since this set of OPEs uniquely defines the entire $W_{1+\infty}$ algebra, it follows that the quantum currents whose leading orders are defined in eqs. (3.1) and (3.3) will generate $W_{1+\infty}$.

A more familiar realisation of $W_{1+\infty}$ is provided by taking a single complex fermion ψ, and building currents by considering all possible bilinear operators with arbitrary numbers of derivatives on the fields [26]. This was originally described in the basis for $W_{1+\infty}$ that corresponds to taking $\alpha = 0$ in the above discussion. Here, we shall describe it for the general case. In fact, the currents that we have constructed above are nothing but the bosonisation of the currents of the fermionic realisation. The bosonisation of the fermionic realisation of $W_{1+\infty}$ was first carried out in ref. [27]. To describe this for arbitrary α, we begin by setting $\hbar = 1$ for convenience. Then the real scalar field φ and the complex fermion ψ are related by

$$\psi = :\mathrm{e}^{\varphi}:, \qquad \bar{\psi} = :\mathrm{e}^{-\varphi}:, \tag{3.6}$$

where :: denotes normal ordering with respect to the modes of φ.

As shown in ref. [27], the fermion bilinear term $\partial^i\bar{\psi}\,\partial^j\psi$ can be expressed as

$$:\partial^i\bar{\psi}(z)\,\partial^j\psi(z): = \sum_{k=i+1}^{i+j+1} \frac{1}{k}(-)^{k-1-i}\binom{l}{k-1-i}\partial^{i+j-k+1}P^{(k)}(z), \tag{3.7}$$

where $P^{(k)}(z)$ is given by

$$P^{(k)}(z) = :\mathrm{e}^{-\varphi(z)}\,\partial^k\,\mathrm{e}^{\varphi(z)}:. \tag{3.8}$$

The currents of $W_{1+\infty}$ in the basis corresponding to an arbitrary value of the parameter α are then given by [25]

$$V^i = \sum_{j=0}^{i+1} a_j(i,\alpha)\,\partial^j\bar{\psi}\,\partial^{i+1-j}\psi, \tag{3.9}$$

where the coefficients $a_j(i,\alpha)$ are given by

$$a_j(i,\alpha) = \binom{i+1}{j}\frac{(i+2\alpha+2-j)_j(2\alpha-i-1)_{i+1-j}}{(i+2)_{i+1}}. \tag{3.10}$$

Here, $(a)_n \equiv (a+n-1)!/(a-1)!$.

Some examples of the fermionised form of the currents for the first few spins are

$$\begin{aligned}
V^{-1} &= \bar{\psi}\psi,\\
V^0 &= \left(\alpha+\tfrac{1}{2}\right)\partial\bar{\psi}\,\psi + \left(\alpha-\tfrac{1}{2}\right)\bar{\psi}\,\partial\psi,\\
V^1 &= \tfrac{1}{3}\left(\alpha+\tfrac{1}{2}\right)(\alpha+1)\,\partial^2\bar{\psi}\,\psi + \tfrac{2}{3}(\alpha+1)(\alpha-1)\,\partial\bar{\psi}\,\partial\psi + \tfrac{1}{3}\left(\alpha-\tfrac{1}{2}\right)(\alpha-1)\bar{\psi}\,\partial^2\psi.\\
V^2 &= \tfrac{1}{60}(2\alpha+3)(2\alpha+1)(\alpha+1)\,\partial^3\bar{\psi}\,\psi + \tfrac{1}{20}(2\alpha+3)(2\alpha-3)(\alpha+1)\,\partial^2\bar{\psi}\,\partial\psi\\
&\quad + \tfrac{1}{20}(2\alpha+3)(2\alpha-3)(\alpha-1)\,\partial\bar{\psi}\,\partial^2\psi + \tfrac{1}{60}(2\alpha-3)(2\alpha-1)(\alpha-1)\bar{\psi}\,\partial^3\psi.
\end{aligned} \tag{3.11}$$

In the bosonised form, these become

$$
\begin{aligned}
V^{-1} &= \partial\varphi\,,\\
V^{0} &= \tfrac{1}{2}(\partial\varphi)^2 + \alpha\,\partial^2\varphi\,,\\
V^{1} &= \tfrac{1}{3}(\partial\varphi)^3 + \alpha\,\partial\varphi\,\partial^2\varphi + \tfrac{1}{3}\alpha^2\partial^3\varphi\,,\\
V^{2} &= \tfrac{1}{4}(\partial\varphi)^4 + \alpha(\partial\varphi)^2\,\partial^2\varphi + \tfrac{1}{20}(8\alpha^2-3)(\partial^2\varphi)^2\\
&\quad + \tfrac{1}{10}(4\alpha^2+1)\,\partial\varphi\,\partial^3\varphi + \tfrac{1}{60}\alpha(4\alpha^2+1)\,\partial^4\varphi\,.
\end{aligned}
\tag{3.12}
$$

We are now in a position to make some comparisons with the results of sect. 2, where we showed how the process of adding counterterms and modifying the transformation rules could be used in order to eliminate the matter-dependent anomalies arising from quantising the original w_∞ gravity theory. The counterterms give rise to quantum corrections to the currents. For spins 2 and 3, the full quantum-corrected currents were given by eqs. (2.12) and (2.13). For spin 4, the $O(\hbar)$ corrections were given by the counterterm (2.18). After setting $\hbar = 1$ for the purposes of making comparisons with our results in this section, we see that the quantum-corrected currents in sect. 2 correspond to our results (3.12) with the parameter α chosen to be

$$\alpha = \tfrac{1}{2}\,. \tag{3.13}$$

The significance of the value $\frac{1}{2}$ for the parameter α is that for this value only, one can truncate out the spin-1 current V^{-1} (and no others) from the $W_{1+\infty}$ algebra, in order to obtain W_∞ [24, 25]. In the equivalent description in sect. 2, this corresponds to the fact that we were able to renormalise the transformation rules and currents of the original classical w_∞ gravity, without needing to add any other fields or currents. This contrasts with an approach discussed in ref. [17], where, starting from W_3 gravity, an infinity of additional currents and fields were introduced in order to close the algebra at the quantum level.

4. The universal anomalies

The W_∞ algebra, given in terms of the operator-product expansions for the currents of sect. 3 (with $\alpha = \frac{1}{2}$), takes the form

$$V^i(z)V^j(w) \sim -\sum_l f^{ij}_{2l}(\partial_z, \partial_w)\frac{V^{i+j-2l}(w)}{z-w} - c_i\delta^{ij}\partial_z^{2i+3}\frac{1}{z-w}\,, \tag{4.1}$$

where $c_i = 2^{2i-3} i!(i+2)! c((2i+1)!!(2i+3)!!)^{-1}$ and c is the central charge [5]. The quantities $f_{2l}^{ij}(m,n)$ are related to the structure constants of the W_∞ algebra precisely as described in subsect. 2.2 of the second paper in ref. [5]. Specifically, we have

$$f_{2l}^{ij}(m,n) = \frac{\phi_{2l}^{ij}}{2(2l+1)!} M_{2l}^{ij}(m,n), \tag{4.2}$$

where

$$\phi_{2l}^{ij} = {}_4F_3\left[\begin{matrix} -\frac{1}{2}, \frac{3}{2}, -l-\frac{1}{2}, -l \\ -i-\frac{1}{2}, -j-\frac{1}{2}, i+j-2l+\frac{5}{2} \end{matrix} ;1\right] \tag{4.3}$$

and

$$M_{2l}^{ij}(m,n) = \sum_{k=0}^{2l+1} (-)^k \binom{2l+1}{k} (2i-2r+2)_k [2j+2-k]_{2l+1-k} m^{2l+1-k} n^k, \tag{4.4}$$

with $(a)_n \equiv (a+n-1)!/(a-1)!$ and $[a]_n \equiv a!/(a-n)!$.

The gauge fields A_i must also transform so as to give a realisation of the W_∞ algebra. Since the currents transform under the adjoint representation of the algebra, it follows that the gauge fields must transform under the coadjoint action. The procedure for calculating these transformation rules was described in detail in ref. [28], where it was shown that the gauge-transformation rules for W_∞ are

$$\delta A_i = \bar{\partial} k_i + \sum_{l \geq 0} \sum_{j=0}^{i+2l} f_{2l}^{j,i-j+2l}(\partial_A, \partial_k) A_j k_{i-j+2l}. \tag{4.5}$$

The notation $f_{2l}^{ij}(\partial_A, \partial_k)$ in eq. (4.5) indicates that the m and n arguments in $f_{2l}^{ij}(m,n)$ are to be replaced by partial derivatives acting either on A only, or on k only. One can easily check that the leading-order ($l=0$) terms in eq. (4.5) coincide with the classical w_∞ transformations given in eq. (2.3). The renormalisation derived in eq. (2.17) from our analysis of the anomalies arising from the Feynman diagram shown in fig. 2 is a special case of the general result (4.5) at the $l=1$ level.

We have shown that the full lagrangian for w_∞ gravity, consisting of the classical term (2.1) together with all counterterms necessary for the cancellation of all matter-dependent anomalies, must be given by

$$L = \tfrac{1}{2} \bar{\partial}\varphi \, \partial\varphi - \sum_{i \geq 0} A_i V^i(\varphi), \tag{4.6}$$

where $V^i(\varphi)$ are the bosonic W_∞ currents in sect. 3 with α set equal to $\frac{1}{2}$. Note that the renormalised currents in the lagrangian (4.6) are to be considered as functions of the scalar field φ, as appropriate for path-integral quantisation. Since these currents contain quantum corrections needed to remove the matter-dependent anomalies, (4.6) is not itself classically W_∞ invariant. Thus, one cannot derive the full quantum-corrected φ-field transformations from it by requiring invariance. Instead, one must consider the renormalised currents $V^i(\varphi)$ as quantum operators and derive the φ-transformations either by taking quantum commutators or, equivalently, by using eq. (2.14). The full quantum-corrected transformation rules for the gauge fields are given by eq. (4.5).

At this stage, we have essentially completed our demonstration that the quantisation procedure, driven by the necessity to remove matter-dependent anomalies, requires the renormalisation of w_∞ into W_∞. The discussion of the closure of the current algebra given in sect. 3 is in a one-to-one relation with the discussion of the renormalisations necessary to remove the matter-dependent anomalies. From the explicit analysis that we have carried out, it seems clear that this complementarity of the two discussions will continue to all orders. Presumably, one can construct a formal proof that the removal of matter-dependent anomalies is always equivalent to the achievement of a closed quantum algebra of currents. Such a proof would require careful analysis of the general anomalous Ward identities including both diagrams with external matter lines and with external gauge-field lines. In the present case, however, we have another way to demonstrate conclusively that the classical lagrangian plus counterterms (4.6) leads to a theory in which all matter-dependent anomalies are cancelled. This is the existence of the fermionic equivalent to our effective action, with the currents given in eqs. (3.9) and (3.10) (with $\alpha = \frac{1}{2}$ in order to truncate out the spin-1 current). In the fermionic realisation, the whole effective action arises from the tree and one-loop orders, and there are clearly no matter-dependent anomalies. Bosonisation of this quantum system yields the theory whose classical action plus counterterms are given in eq. (4.6).

With the matter-dependent anomalies out of the way, we can now focus attention on the universal anomalies. The derivation given below is similar to one given in ref. [29] for W_3 gravity. In order to derive an anomalous Ward identity for these, we are now free to make use of the current algebra (4.1) and the coadjoint gauge field transformations (4.5). Considering only diagrams with external gauge-field lines, the effective action is, in terms of operator expectation values.

$$e^{-\Gamma(A_i)} = \left\langle \exp\left(-\frac{1}{\pi} \int \sum_i A_i V^i \right) \right\rangle, \tag{4.7}$$

which can also be written in path-integral language as

$$e^{-\Gamma(A_i)} = \int \mathscr{D}\varphi \exp\left(\frac{1}{\pi} \int \left(\frac{1}{2} \bar{\partial}\varphi\, \partial\varphi - \sum_i A_i V^i \right) \right). \tag{4.8}$$

Varying (4.7) with respect to $A_i(z)$, one finds

$$\frac{\delta\Gamma}{\delta A_i(z)}=\frac{1}{\pi}\left\langle V^i(z)\exp\left(-\frac{1}{\pi}\int\sum_j A_jV^j\right)\right\rangle e^{\Gamma} \tag{4.9}$$

and hence

$$\bar\partial\frac{\delta\Gamma}{\delta A_i(z)}=\frac{1}{\pi}\left\langle \bar\partial V^i(z)\exp\left(-\frac{1}{\pi}\int\sum_j A_jV^j\right)\right\rangle e^{\Gamma}. \tag{4.10}$$

The occurrence of the $\bar\partial=\partial_{\bar z}$ derivative in eq. (4.10) means that the only non-zero contributions will come from $\bar\partial$ acting on singular terms in the operator product expansion of the operator being averaged. Thus, we may calculate

$$\begin{aligned}&\bar\partial V^i(z)\exp\left(-\frac{1}{\pi}\int\sum_j A_jV^j\right)\\&\quad=\bar\partial V^i(z)\sum_{n\geqslant 0}\frac{1}{n!}\left(-\frac{1}{\pi}\int A_jV^j\right)^n\\&\quad=-\frac{1}{\pi}\sum_{n\geqslant 1}\frac{1}{(n-1)!}\int d^2w\left[\bar\partial V^i(z)V^j(w)\right]A_j(w)\left(-\frac{1}{\pi}\int A_kV^k\right)^{n-1},\end{aligned} \tag{4.11}$$

where the brackets around $\bar\partial V^i(z)V^j(w)$ in eq. (4.11) indicate that the operator-product expansion should be taken just between these two operators.

Using eq. (4.1), the operator products in eq. (4.11) may be evaluated to give

$$\begin{aligned}\bar\partial V^i(z)\exp\left(-\frac{1}{\pi}\int\sum_i A_iV^i\right)&=\frac{1}{\pi}\int d^2w\,\partial_{\bar z}\left(\sum_{l\geqslant 0}f^{ij}_{2l}(\partial_z,\partial_w)\frac{V^{i+j-2l}(w)}{z-w}+c_i\partial_z^{2i+3}\frac{1}{z-w}\right)\\&\qquad\times A_j(w)\exp\left(-\frac{1}{\pi}\int A_kV^k\right).\end{aligned} \tag{4.12}$$

Since $\partial_{\bar z}1/(z-w)=\pi\delta^{(2)}(z-w)$, we may perform the integration in (4.12). Thus we find, from eq. (4.10), that

$$\bar\partial\frac{\delta\Gamma}{\delta A_i}+\sum_{l\geqslant 0}f^{ij}_{2l}(\partial,-\partial_A)\left(\frac{\delta\Gamma}{\delta A_{i+j-2l}}A_j\right)=-\frac{c_i}{\pi}\partial^{2i+3}A_i. \tag{4.13}$$

The subscript A on the second derivative argument of f^{ij} indicates that it should act only on the explicit A_j in the parentheses that follow it, whilst the first derivative argument of f^{ij} acts on all terms in the parentheses. Eq. (4.13) is the anomalous Ward identity for W_∞ gravity.

If we now multiply by the spin-$(i+2)$ transformation parameters k_i and integrate, we find

$$\int \frac{\delta\Gamma}{\delta A_i}\left(\bar{\partial}k_i + \sum_{l\geq 0}\sum_{j=0}^{i+2l} f_{2l}^{i-j+2l,j}(\partial_k,\partial_A)k_{i-j+2l}A_j\right) = \frac{c_i}{\pi}\int k_i\partial^{2i+3}A_i. \quad (4.14)$$

From eq. (4.5) we see that the left-hand side of this equation involves precisely the W_∞ gauge-transformation rule for A_i under k_j, and so eq. (4.14) can be written simply as

$$\delta_k\Gamma = \sum_{i\geq 0}\frac{c_i}{\pi}\int d^2z\, k_i\partial^{2i+3}A_i. \quad (4.15)$$

Thus we see that the effective action is not invariant under spin-$(i+2)$ W_∞ transformations, on account of the anomalous terms on the right-hand side, which arise from the central charges in the theory. This result for the universal anomalies is exact to all orders in perturbation theory.

5. Ghosts and universal anomaly cancellation

We have now established that by starting from the single-scalar realisation of w_∞, and renormalising the currents and transformation rules in order to remove the matter-dependent anomalies, we are left with a quantum theory with a W_∞ symmetry, and W_∞ anomalies only in the universal sector. These anomalies, involving only external background gauge fields, are given by eq. (4.15). The coefficient c_i in the spin-$(i+2)$ sector is precisely the spin-$(i+2)$ central charge in the W_∞ algebra [5], as given below eq. (4.1). The $i=0$ case corresponds to the familiar Virasoro anomaly of two-dimensional quantum gravity.

In two-dimensional quantum gravity, one way in which the anomaly can be removed is by including the Virasoro ghosts arising from integration over the two-dimensional metric, in the special case where the matter realisation of the Virasoro algebra has central charge $c=26$. This is the case for the critical bosonic string in 26 dimensions. For W_N gravity, integration over all the higher-spin gauge fields will give a total ghost contribution in the Virasoro sector of

$$c_{gh} = \sum_{s=2}^{N} c_{gh}(s), \quad (5.1)$$

where

$$c_{gh}(s) = -2(6s^2 - 6s + 1) \quad (5.2)$$

is the contribution from the ghosts for the spin-s gauge fields. Thus eq. (5.1) becomes

$$c_{\text{gh}} = -(N-1)(4N^2+4N+2). \tag{5.3}$$

In addition, there will also be ghost contributions to the anomalies in all the higher-spin sectors. Of course the values of the central-charge contributions in the various spin sectors are all related to one another, since there is just one overall central-charge parameter in the W_N algebra.

Naïvely, by setting $N=\infty$ in eq. (5.3), one would think that the total ghost contribution in the Virasoro sector of W_∞ would be $c_{\text{gh}} = -\infty$. However, as discussed in refs. [21, 22], it appears that it is more appropriate to treat the divergent sum (5.1) over the individual spin-s contributions as a quantity that should be rendered finite by some regularisation procedure. Likewise, the ghost contributions in all the higher-spin sectors will be given by divergent sums, which can also be regularised. The regularisation procedures for each spin must be consistent with one another, since there is just one overall central-charge parameter in the W_∞ algebra [5]. In refs. [21, 22] it was shown that a natural zeta-function regularisation scheme gives the regularised result

$$c_{\text{gh}} = 2. \tag{5.4}$$

A consistent extension of this regularisation scheme to all spin sectors was proposed in ref. [22], where it was shown that it gave consistent results at least up to the spin-18 level. The fact that such a universal scheme exists is highly suggestive of an underlying interpretation and rigourous justification for the regularisation procedure, possibly in terms of a higher-dimensional theory [21, 22].

The regularised value of $c_{\text{gh}} = 2$ is exactly what is needed in order to cancel the universal anomalies (4.15). To see this, we note that the single-scalar matter realisation that we are using has a background-charge parameter α (see eq. (3.12)) that must be chosen to equal $\frac{1}{2}$ in order to avoid the occurrence of a spin-1 current in the algebra. Thus the matter field contributes a central charge

$$c = 1 - 12\alpha^2 = -2, \tag{5.5}$$

which is precisely cancelled by the regularised ghost contribution. Thus we see that the total (regularised) universal anomaly for our single-scalar realisation of W_∞ gravity vanishes.

6. A ghost realisation for quantum w_∞ gravity

The reason why the classical w_∞ symmetry of the theory that we have been discussing so far became deformed to W_∞ at the quantum level was that the w_∞

currents only formed a closed algebra at the classical level. If one had a realisation of w_∞ in terms of currents that still closed at the quantum level, then there would be no matter-dependent anomalies, and the currents would not suffer renormalisation. Such a realisation in fact exists. As shown in ref. [22], the currents

$$v^i(z) = \sum_{j \geq 0} \left((i+j+2)\, \partial c_j\, b_{i+j} + (j+1) c_j \partial b_{i+j} \right), \tag{6.1}$$

where $b_i(z)$ and $c_i(z)$ are anticommuting ghosts satisfying

$$b_i(z) c_j(w) \sim \frac{\delta_{ij}}{z-w}, \tag{6.2}$$

generate the w_∞ algebra

$$v^i(z) v^j(w) \sim \frac{i+j+2}{(z-w)^2} v^{i+j}(w) + \frac{i+1}{z-w} \partial v^{i+j}(w) + \delta^{i0}\delta^{j0} \frac{c/2}{(z-w)^4}. \tag{6.3}$$

The operator terms on the right-hand side come from single contractions. The central term in the spin-2 sector (the only one that occurs in the w_∞ algebra) has a central charge that is formally divergent (as described in the previous section). After zeta-function regularisation, one finds $c = 2$.

Since this realisation of w_∞ is linear, all Feynman diagrams will be one-loop, and there are no matter-dependent anomalies. The only anomaly in the theory is the spin-2 universal anomaly. (It is easy to see from the form of the currents (6.1) that no one-loop diagram with any external gauge fields of spins greater than 2 can be constructed.) One could cancel the universal anomaly by introducing a β_i, γ_i bosonic ghost system. The theory that is obtained by this means is precisely topological w_∞ gravity [30].

7. Discussion

In this paper we have shown the sense in which the W_∞ algebra can be considered as a quantum deformation of the w_∞ algebra. We started by quantising a single-scalar realisation of w_∞ gravity. The cancellation of the matter-dependent anomalies necessitates the introduction of an infinite number of counterterms. It turns out that these counterterms correspond exactly to a renormalisation of the classical w_∞ currents to the quantum W_∞ currents.

An interesting feature that we encountered in the quantisation was the necessity of introducing local counterterms containing half-integer powers of Planck's constant $\hbar$. The variation of these counterterms does not give rise to half-integer powers of $\hbar$, but they do give rise to terms proportional to integer powers of $\hbar$ which are crucial in the cancellation of the matter-dependent anomalies. In particular, we find a quantum correction to the energy–momentum tensor of the form $\sqrt{\hbar}\,\partial^2\varphi$. This term has the interpretation as a background charge for the single scalar φ.

The classical w_∞-invariant theory (2.1) that we started from possessed more symmetry than just w_∞. As was shown in ref. [9], this classical action possessed also an infinite set of local Stueckelberg-type symmetries* that cause the gauge fields A_l with $l \geqslant 1$ to shift:

$$\delta A_0 = -\sum_{l \geqslant 1} \frac{2}{l+2} \alpha_l (\partial_+ \varphi)^l ,$$

$$\delta A_l = \alpha_l , \qquad l \geqslant 1 . \tag{7.1}$$

Owing to these symmetries, it is possible to eliminate classically all higher-spin gauge fields A_l with $l \geqslant 1$ in this one-scalar model. As a consequence of this, the w_∞ transformations acquire compensating terms and the whole set of local w_∞ transformations may be viewed as local Virasoro transformations with field-dependent coefficients [9]. (This is the "telescoping" procedure that was discussed in ref. [9].)

It is not obvious what the quantum fate of the Stueckelberg symmetries is. Since the transformations (7.1) involve matter fields as well as the background gauge fields, it would appear that a proper discussion of these symmetries could only be given by treating both kinds of fields on an equal footing. Clearly, the finite local counterterms that we have introduced to cancel the matter-dependent anomalies violate the classical Stueckelberg symmetries given above. This situation may be understood by noting that the Stueckelberg symmetries of (7.1) arise from the fact that the classical higher currents V^i given in eq. (1.4) can be written for $i \geqslant 1$ as the product of the spin-2 current times other currents. It is natural to enquire whether a similar factorisation is possible for the quantum currents. The following two issues arise here. First of all one has to define what one means by the product of two quantum currents. This requires the use of a specific regularisation scheme. It matters now whether one uses a regularisation with respect to the currents or with respect to the fundamental scalar field. The current regularisation scheme is usually adopted in discussions of quantum W-algebras. From the field-theoretic point of view, however, it is necessary to use a scalar field regularisation. Secondly, in order for the Stueckelberg symmetries to occur, it is sufficient that the higher

* The w_∞ gravity theory also has further symmetries, called β and γ symmetries in ref. [9]. They seem not to play an important rôle in the quantisation of the theory.

currents factorise into the product of a lower-spin currents times terms containing the scalar field. From a field-theoretic point of view it does not seem necessary to require that the extra terms containing the scalar field can be written in terms of currents too, at least if one uses a regularisation with respect to the scalar field.

As an example, we consider the possible factorisations of the spin-4 current V^2. Applying (3.12) for $\alpha = \frac{1}{2}$, we find that V^2 is given (with $\hbar$ set equal to 1) by

$$V^2 = \tfrac{1}{4}(\partial\varphi)^4 + \tfrac{1}{2}(\partial\varphi)^2\partial^2\varphi - \tfrac{1}{20}(\partial^2\varphi)^2 + \tfrac{1}{5}\partial\varphi\,\partial^3\varphi + \tfrac{1}{60}\partial^4\varphi. \tag{7.2}$$

Using a regularisation with respect to φ, one can rewrite V^2 as follows:

$$V^2 = V^0V^0 + \tfrac{1}{30}\partial^2V^0 + \tfrac{1}{3}\partial\varphi\,\partial V^0 - \tfrac{2}{3}\partial^2\varphi\,V^0. \tag{7.3}$$

A similar factorisation can be carried out for all currents V^i with $i \geqslant 1$.

There exists another factorisation of the spin-4 current which uses a current regularisation. To make contact with the standard formulation of the quantum W_3 algebra we present this factorisation as well. In general, for currents $A(z)$ and $B(z)$ the normal-ordered product with respect to the modes of A and B is defined by

$$(AB)(w) = \oint \frac{dz}{2\pi i}\,\frac{A(z)B(w)}{z-w}. \tag{7.4}$$

Using this regularisation, the spin-4 current can be rewritten as

$$V^2 = (V^0V^0) - \tfrac{3}{10}\partial^2V^0. \tag{7.5}$$

This relation can be used at the right-hand side of the OPE of V^1 with V^1 to express the spin-4 current as a composite in terms of the spin-2 current. This gives rise to the usual formulation of W_3 as a closed, but nonlinear algebra. This single-scalar quantum realisation of W_3, with central charge $c = -2$, was first constructed in ref. [13].

Going back to the Stueckelberg symmetries, it was shown in ref. [9] that a realisation of classical w_∞ gravity in terms of $(N-1)$ scalars corresponding to the Cartan subalgebra of SU(N) leads to a theory for which Stueckelberg symmetries can be used to eliminate only gauge fields with spin greater than N. In the full quantum theory, in which gauge fields as well as matter fields are quantised, it is conceivable that there may be some quantum analogue of the Stueckelberg symmetries, that could be used to eliminate some of the gauge fields. Thus it is natural to consider models in which the gauge fields with spins less than or equal to N are protected from elimination by Stueckelberg symmetries by considering an $(N-1)$-scalar realisation of W_∞. Since the algebra is linear, the easiest way to obtain such a realisation is to take $(N-1)$ copies of the single scalar realisation:

$$V^i = \sum_{\alpha=1}^{N-1} V^i(\varphi_\alpha), \tag{7.6}$$

where $V^i(\varphi_\alpha)$ denotes the currents for the scalar field φ_α. The renormalisation of the currents, for each of the scalars φ_α, will proceed precisely in the same way as we have described in detail for the one scalar case. It would be interesting to see whether one could obtain quantum W_N gravity by a telescoping procedure from quantum W_∞ gravity.

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